Lesson 3. Introduction to Stochastic Processes and Markov Chains

1 What is a stochastic process?

- A **stochastic process** is a mathematical model of a probabilistic system that evolves over time and generates a sequence of numerical values
    - Each numerical value in the sequence is modeled by a random variable
- Examples:
    - the sequence of daily prices of a stock
    - the sequence of failure times of a machine
    - the sequence of radar measurements of the position of an airplane
- Another perspective: a stochastic process is a sequence of random variables ordered by an index set
    - The indices are often referred to as "time" or "time steps"
- Examples:
    - $\{S_n : n = 0, 1, 2, \dots\} = \{S_0, S_1, S_2, \dots\}$ with discrete index set $\{0, 1, 2, \dots\}$ is a **discrete-time process**
    - $\{Y_t : t \geq 0\}$ with continuous index set $\{t \geq 0\}$ is a **continuous-time process**
- The **state space** of a stochastic process is the range (possible values) of its random variables
    - State spaces can be discrete or continuous
      (i.e. the random variables of a stochastic process can be discrete or continuous)

2 Submarine behavior, revisited

Recall the setting from Lesson 1:

> Ballistic missile submarines act as a nuclear deterrence to enemy countries. These submarines move randomly throughout the ocean within a fixed grid assigned by a higher authority. Approximately every 20 minutes, the submarine will turn to change its course in order to clear its *baffles*, the area directly behind the submarine where sonar cannot detect sound.

Let's consider a similar example, in which a single submarine moves between cells in a 2×2 grid:

1	2
3	4

As before, we model each time step as a 20 minute interval. The submarine starts in cell 1. At each time step, the submarine randomly moves to an adjacent cell. The submarine's next location depends only on its current location, according to the following probabilities:

		ending cell			
		1	2	3	4
starting cell	1	0	0.95	0.01	0.04
	2	0.27	0	0.63	0.10
	3	0.36	0.40	0	0.24
	4	0.11	0.71	0.18	0

3 Markov chains

- Discrete-time, discrete-state stochastic process $\{S_0, S_1, S_2, \dots\}$ where

$$S_n = \textbf{state} \text{ at time step } n$$
$$\mathcal{M} = \{1, \dots, m\} = \textbf{state space}\text{, or set of possible states}$$

- $\{S_0, S_1, S_2, \dots\}$ is a **Markov chain** if:

    - In other words, $\{S_0, S_1, S_2, \dots\}$ satisfies **the Markov property**: the conditional probability of the next state given the history of past states only depends on the last state

- States evolve according to **transition probabilities**:

- The initial state S_0 is determined by the **initial state probabilities**:

Example 1. Compute the probability that the submarine moves from cell 1 to cell 2 to cell 4.

- The sample paths of a Markov chain are completely characterized by a corresponding sequence of transition probabilities and initial state probabilities

- Note that a Markov chain is **time stationary** because

    - In other words, the conditional probability of the next state given the last one does not depend on the number of time steps taken so far

- Sometimes it is useful to study a variation of the Markov chain that is not time stationary, where the transition probabilities may change over time

    - These kinds of Markov chains are beyond the scope of this course

4 Representations of Markov chains

- We can organize the step transition probabilities into a **transition probability matrix**:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

- We can also organize the initial state probabilities into a **initial state vector**:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}$$

Example 2. Write the transition probability matrix and initial state vector for the submarine Markov chain. Why do the rows of the transition probability matrix sum up to 1?

- We can also draw a **transition probability diagram** where
  - each node represents a state of the system
  - a directed arc connects state i to state j if a transition from i to j is possible in one step
  - the transition probability p_{ij} is written next to the arc from i to j

Example 3. Draw the transition probability diagram for the submarine Markov chain.

5 Next time...

- How can we use the transition probability matrix $\mathbf{P}$ and initial state vector $\mathbf{q}$ to answer questions like:
  - Given that we are in state i right now, what is the probability we will be in state j after n time steps?
  - What is the unconditional probability we will be in state j after n time steps?

6 Exercises

Problem 1 (SMAS Exercise 6.4). Consider a Markov chain with state space $\mathcal{M} = \{1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{bmatrix}$$

Compute the following:

a. $\Pr\{S_1 = 2 \mid S_0 = 1\}$
b. $\Pr\{S_2 = 1 \text{ and } S_1 = 1 \mid S_0 = 2\}$
c. $\Pr\{S_2 = 1 \mid S_0 = 1\}$

Problem 2 (SMAS Exercise 6.5). Consider a Markov chain with state space $\mathcal{M} = \{1, 2, 3, 4\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0.4 & 0.1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Draw the transition probability diagram.